Evaluation of Structural Damping by the Fourier Coefficients of the Modulated Steady-State Responses

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Abstract

The present article gives a novel method for evaluating the viscous damping ratios of single-degree-of-freedom systems. Unlike the most existing methods that are based on the system transient responses and/or the assumption of small damping, the present method is derived from the steady state. In other words, the present method takes the input signals as the reference and the steady-state output as the modulator. By applying the external excitation at the particular frequency, one is then to transform the steady-state signal to the frequency domain and find its Fourier Coefficients. The system damping ratio can be expressed as the function of the ratio of these Fourier Coefficients, through the tangent of the lag angle. It is also the reason the present method is named as the 'Fourier Coefficient Ratio' method (FCR). The new FCR method is further verified by numerical simulations in this report. In addition, a cantilever beam with an end damper system was designed to experimentally test the method. Not only have the results from numerical simulations but from the experiments as well all substantiate the validity of the present method.

keywords: Damping ratio, Identification of Damping, Viscous Damping.

1. Introduction

Unlike mass and stiffness properties, which can be directly measured or analytically predicted by numerical models such as the finite element, the damping of system is extremely difficult to obtain. Generally, the damping characteristics, no matter they are internal (material, micro-structural effects, friction, etc.) or external (boundary, fluid contact, fluid/structure interaction, etc.), can be revealed only by experimental measurement. Therefore, reasonably accurate identification methods that correlate the analytical model with measured data not only are important but necessary as well. In fact, there exist many models being applied in various fields. For example, Ibrahim [1,2], Armstrong-Helouvry [3] have provided detail surveys from the viewpoint of models, control and compensation. Even with such models, one still has to measure the system damping or dissipative energy in order to adequately apply them to the real system.

In order to estimate or measure the system damping (or damping ratio, ς), there are several means have been developed. Some are well documented in (e.g., Jones [4], Macconnell [5], Lamarque, et. al [6]) for various systems. Among them, the most commonly seen method is the so-called 'Q-factor' method. That is, one calculates the half power bandwidth ($\Delta\omega$) of the two half-power points, which locate at the either side of the resonant peak in the frequency domain. By computing the ratio, or called the Q-factor, of the central resonant frequency to $\Delta\omega$, one is able to estimate the damping ratio. However, the validity of the relation between the Q factor and the system damping ratio is limited to systems with small damping. That is, the method cannot be used in case Q-factor exceeds 20 or so in order to get sufficiently accurate results (Woodhouse [7]). This value is equivalent to the damping ratio below around 0.05.

In addition to the mentioned Q-factor method, the 'logarithmic decrement' method is also well-known. Basically, the logarithmic decrement is defined by the successive peak-amplitudes of the decaying responses. By measuring the amplitude ratio of the decaying responses, one is able to compute the corresponding damping ratio from that time domain responses. Again, similar to the former method, this evaluating method is only good for small damping systems.

Recently, the first author has derived a new idea [8] for measuring structural damping. In that method, the response modulation has been first applied. The idea is not only good for small damping systems, but large ones as well. However, it needs to measure three parameters in order to evaluate the system damping ratio. Therefore, the method reported [8] is too difficult to accurately measure the true damping. Motivated by this, the present research tries to bridge the gap by minimizing the number of parameters involved in the evaluating process. However, the theoretical back-ground has to be introduced first.

2. Derivation of the method

2.1 Direct Excitations

The responses of a linear SDOF system under an external excitation can be represented by

$$m\ddot{y} + c\dot{y} + ky = f(t) \quad (1)$$

where m, c, and k are the mass, damping and stiffness, respectively, $f(t)$ the externally applied forcing function. Or, equivalently, equation (1) can be also expressed as

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = x(t) \quad (2)$$

where

$$\omega_n^2 = \frac{k}{m} \text{ and } \zeta = \frac{c}{2\sqrt{mk}}. \quad (3)$$

The solution of equation (2) can be written as

$$y(t) = y_0(t) + y_p(t) \quad (4)$$

in which $y_0(t)$ is the free or transient response and characterized by the system itself and the initial conditions, and $y_p(t)$ the forced or steady-state response. It is also evident that the latter mainly depends on the excitation. And, in case the excitation is harmonic, or

$$f(t) = F\sin(\Omega t), \quad (5)$$

then $y_p(t)$ can be represented as

$$y_p(t) = Y\sin(\Omega t - \phi) \quad (6)$$

where

$$Y = \frac{X}{\sqrt{(1-r^2)^2 + (2r\zeta)^2}} \quad (7)$$

$$X = F/k, \text{ and } \tan\phi = \frac{2r\zeta}{1-r^2}. \quad (8)$$

And, the frequency ratio is defined as

$$r = \Omega/\omega_n. \quad (9)$$

All these equations are quite fundamental and can be easily found in vibration textbooks. However, the natural frequency of a system is in general unknown, eq. (9) is actually no much help during a vibration test. Instead, one is able to locate the frequency where the maximum amplitude occurs. Or, the frequency is

$$\omega_m = \omega_n\sqrt{1-2\zeta^2} \quad (10)$$

with $\zeta^2 \le 0.5$ from eq. (7). Notice that the maximum amplitude occurs when $r < 1$ since damping exists. Theoretically, the location of ω_m can be easily found by the frequency response function (FRF, $H(\Omega)$) and noticing the location where the peak amplitude occurs.

In order to measure the system damping ratio, one uses eq. (11) and rewrites eq. (8) in terms of ω_m,or let $r = \sqrt{1-2\zeta^2}$ in eq. (8). This simply means carrying out the experimental measurement right at ω_m. In addition, one can solve for the system damping at that particular situation, or

$$\zeta = \frac{1}{\sqrt{2+G_0^2}} \quad (11)$$

where G_0 is used to signify $\tan\phi$ when the excitation frequency is at the maximum amplitude occurs. Therefore, the core problem now becomes how to evaluate the tangent for the lag angle ϕ, which depends on the system damping. This will be discussed in Section 2.3.

Examining eq. (11), one can clearly find out that the validity of this equation holds without imposing an additional condition like small damping. Instead, refer to eq. (10), the system damping ratio can be as high as 0.707. And, mathematically ζ is defined for all $\tan\phi$ in $(-\infty, +\infty)$, refer to Fig. 1. However, since eq. (11) requires $\Omega = \omega_m$, $\tan\phi$ (or G_0) is thus limited in the positive, as shown in Fig. 1

Examining eq. (11), one may find that the system damping ratio is inverse proportional to the value of G_0. This is because that the location where the maximum amplitude occurs close to but less than the resonance frequency, or at $r < 1$. As the consequence, the response always lags the excitation at that frequency ω_m. Meanwhile, G_0 takes only positive value.

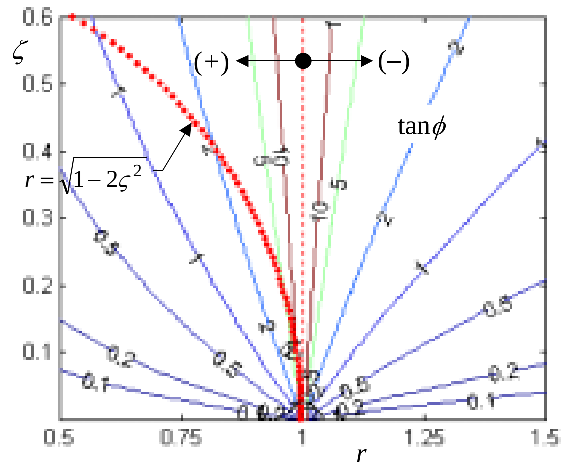

Fig. 1 Damping ratio ζ vs. r and $\tan\phi$.

2.2 Support Excitations

Similar to the former situation, in case the SDOF system is excited through its support by $u(t)$, the absolute response of the mass, $y(t)$, is governs by the equation of motion [9]

$$m\ddot{y} + c(\dot{y} - \dot{u}) + k(y - u) = 0 \quad (12)$$

or the relative motion $z(t) = y - u$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{u} \quad (13)$$

where m, c, and k are the same as in eq. (1). If the base excitation satisfies

$$u(t) = U\sin\Omega t, \quad (14)$$

the amplitude of the steady-state responses of the mass can be also represented by the form (6) with

$$Y = R_Y \cdot U, \quad (15)$$

and

$$\tan\phi = \frac{2\zeta r^3}{(1-r^2)+(2r\zeta)^2}. \tag{16}$$

Note, R_Y in eq. (15) is called the transmissibility to denote the portion of the input amplitude that has been transmitted to the output. In fact, the transmissibility can be defined as

$$R_Y = \left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}. \tag{17}$$

Again, the maximum amplitude of system in eq. (13) occurs at the circular frequency

$$\omega_m = \frac{\omega_n}{2\zeta}\sqrt{-1+\sqrt{1+8\zeta^2}} \quad \text{for} \quad \zeta > 0 \tag{18}$$

obtained from eq. (17). Following the similar steps given in Section 2.1, $\tan\phi$ can be evaluated at ω_m by substituting eq. (18) back to (16). Therefore, the system damping ratio of (13) can be computed at that particular frequency. However, since the permutation is much tedious than that in Sec. 2.1, one has to utilize a symbolic computation software. In the present study, the commercial package Mathematica® was used. The result for ζ as the function of G_0 can be expressed by

$$\zeta^2 = \frac{1}{8}\left[2+\frac{(G_0^2-8)}{G_0^{2/3}\Lambda}+\frac{\Lambda}{G_0^{4/3}}\right] \quad \text{if} \quad G_0 \le \sqrt{8} \tag{19}$$

or the real part of

$$\zeta^2 = \frac{1}{16}\left[4-(1+i\sqrt{3})\frac{(G_0^2-8)}{G_0^{2/3}\Lambda}-(1-i\sqrt{3})\frac{\Lambda}{G_0^{4/3}}\right] \quad \text{if}$$

$$G_0 > \sqrt{8} \tag{20}$$

where

$$\Lambda = \sqrt[3]{8-20G_0^2-G_0^4+8\sqrt{(1+G_0^2)^3}}\ . \tag{21}$$

Parameter G_0 in eqs. (19) and (20) is used for the shorthand of $\tan\phi$ at ω_m. In addition, only the real, positive root of them is of interest, according the definition of the system damping ratio. Note that the relation in these two equations guarantees the existence of the damping ratio, if $\tan\phi$ at ω_m is known. Therefore, the measurement of the damping ratio again reduces to the evaluation of G_0.

Actually, in order to measurement the damping ratio of a support excited system, the author [10] has developed the other method, which is based on locating the frequency (ω_0) at $\phi = \pi/2$. Thus, these system frequencies satisfy the relation

$$\omega_m < \omega_n \le \omega_0 \tag{22}$$

when the system damping ratio is $0 < \zeta < 1$. In addition, $\tan\phi$ (or G_0) as function of ζ and r (or ω_n) is plotted and shown in Fig. 2 for all r in the vicinity of r = 1. Examining Fig. 2, it can be clearly seen that ω_0 exists only when $r \ge 1.0$, while ω_m for $r < 1.0$. Thus, G_0 at ω_m is always positive. Moreover, ω_m can be found only for a limited range of positive G_0.

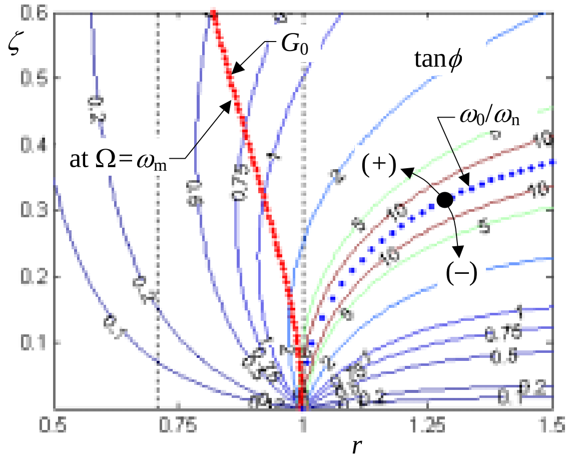

Fig. 2 $\tan\phi$ and G_0 in the vicinity of ω_n (r = 1).

2.3 Evaluation of tanϕ

In order to evaluate the tangent of the lag angle ϕ at a particular external excitation, [8] derived a novel method that using input forcing function and its 90° phase shift as the two reference signals, the system modulated responses are then obtained. By applying two low-pass-filters, the two dc's of the modulated signals are of the main interest. Later, the method was modified [10] in such a way that only one reference input is required, together with the response amplitude lock-in. Therefore, analogous to the same idea that using input as the reference signal, i.e., one starts with defining the response modulation response from eq. (6) by

$$g(t) = f(t)\cdot y(t) = F\sin(\Omega t)\cdot Y\sin(\Omega t-\phi), \tag{23}$$

where $f(t)$ can be either the forcing function in the externally excited system, or the displacement function $u(t)$ in the base excited one.

Clearly, one may also equivalently write eq. (24) as

$$g(t) = \frac{FY}{2}\left[\cos\phi-\cos(2\Omega t-\phi)\right]. \tag{24}$$

Notice that the first term in RHS of eq. (24) is time-invariant, while the second has the frequency of 2Ω. This simply means that the first term of $g(t)$ is dc. Hence, denoting the two Fourier coefficients of $g(t)$ are a_0 and a_2 respectively, one has

$$\cos\phi = \frac{a_0}{a_2} \tag{25}$$

in which the negative sign of a_2 has been omitted since the angle caused by the system damping at ω_m has been known to lag the input. In addition, the cosine of ϕ has been expressed in terms of the two Fourier coefficients' ratio, equation (25) does not depend on either the amplitude of the forcing function or that of the response.

Tangent of ϕ at ω_m can be therefore computed as

$$G_0 = \frac{a_2\sqrt{1-(a_0/a_2)^2}}{a_0}\ . \tag{26}$$

Note that the procedures applying eqs. (24) to (26) are valid for all excitation frequencies. However, in order to comply with eqs. (12), (20) or (21), the excitation frequency must be right at ω_m. Therefore, with the tangent of ϕ obtained, the computation of the damping ratio can be easily done by one of these equations.

The measurement of a_0 may be completed by the hardware electronic circuit through a low pass filter (LPF) as given in [8]

$$a_0 = \|g(t)\|_{\mathrm{LPF}} = \frac{F\cdot Y}{2}\cos\phi\ . \tag{27}$$

Or, one may simply find the time average value of $g(t)$

$$a_0 \cong \frac{1}{n}\sum_{i=1}^{n} g(i\cdot\Delta t_s) \tag{28}$$

with $\Delta t_s = (1/f_s)$ and f_s is the sampling frequency. In addition, examining eq. (24), a_0 can never be negative since $\phi < \pi/2$ at $\Omega = \omega_m$.

On the other hand, the coefficient a_2 can be obtained similar to that of a_0 by a band-pass filter (BPF),

$$a_2 = \|g(t)\|_{\mathrm{BPF}} = -\frac{F\cdot Y}{2} \tag{29}$$

Or, by software that to compute the peak value at 2Ω after the Fourier transform. Figure 3 depicts the schematic procedures of the method. This will be further stressed by a few examples in the following Sections.

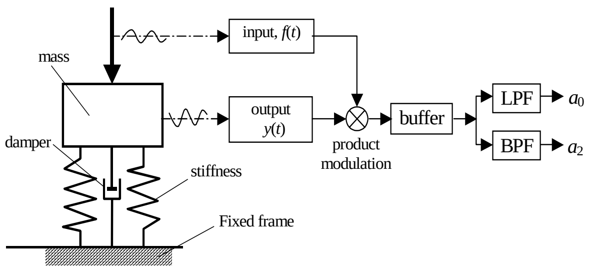

Fig. 3 Schematic block diagram of the method

Notice that $\tan\phi$ at ω_m can be also evaluated from the FRF. That is, let $H(.)$ denote the corresponding FRF of the system, then [e.g., 5, 9]

$$\tan\phi = \left.\frac{\mathrm{Im}[H(\omega)]}{\mathrm{Re}[H(\omega)]}\right|_{\omega=\omega_m}, \tag{30}$$

which can be found from most texts. In addition, as far as instruments are concerned, they all have provided such function as built-in. However, the main problem stems from finding the FRF in eq. (24). Theoretically, it needs a white excitation, which does not exist in the real world, to excite the target system in order to get the exact FRF. An alternative is a wide band excitation which has been used as the substitute of the white one.

2.4 Complete Procedures

In accordance with the mentioned theoretical background, one summarizes the whole procedures for measuring damping ratios:

(1) Locating the frequency (ω_m) where the maximum amplitude occurs. This can be done by applying an impulse or a sweep sine test.

(2) Apply a harmonic excitation to the system with the fixed frequency right at ω_m. Note, the care should be taken since large responses may occur.

(3) Acquire the system steady-state response, together with the input harmonic excitation. The two signals are then used to compute the response product modulation, $g(t)$. Zoom transformation [5] may be required in order to increase the frequency resolution.

(4) Find the Fourier Transform of $g(t)$ as well as its coefficients a_0 and a_2. They can be found by examining at the dc and at two times of ω_m, respectively.

(5) Compute $\tan\phi$ at ω_m (or G_0) using eq. (25) together with eq. (26).

(6) Obtain the system damping ratio using eqs. (11), (19) or (20), respectively, depending the system and G_0.

(7) Verify the result if necessary.

Notice, from step (4) above, the computation of the two Fourier coefficients are required in order to evaluate the structural damping ratio, the present method is thus called the Fourier Coefficient Ratio (FCR) method.

3. Numerical Simulations

In order to verify the validity of the mentioned FCR method, numerical simulations using MATLAB have been conducted in this Section for the case of directly excited systems. And, the experiments will be carried out in the following section for systems under base excitations.

Referring to Fig. 3 and the procedures in Sec. 2.4, the frequency of the system resonance has to determine prior to simulations.

Four different cases have been investigated for various damping. The damping ratios are 0.02, 0.05, 0.1 and 0.5. Among them, the first two are supposed to be ‘small’ damping, in which all existing evaluating method can be applied. In the present simulation, the quality factor (Q-factor) method is adopted for the reason of comparison. Results are shown in Table 1. In addition to the damping ratios, the frequency resolution, Δf, and the standard deviation (s.d.) are also tabulated.

Table 1 Results of numerical simulations

Case	method	ζ (s.d.)	Δ*f* (Hz)
1	True	0.02	N/A
	Q-Factor	0.0200	0.01
	FCR	0.0261(0.0044)	0.009 ~ 0.05
2	True	0.05	N/A
	Q-Factor	0.0500	0.01
	FCR	0.0499(0.0050)	0.009 ~ 0.05
3	True	0.1	N/A
	Q-Factor	0.1020	0.01
	FCR	0.1136(0.0187)	0.01 ~ 0.05
4	True	0.5	N/A
	Q-Factor	0.8244	0.01
	FCR	0.5413(0.0364)	0.02 ~ 0.05

From the results shown in Table 1, one can clearly concludes that both the present method and the Q-factor method predict the corresponding true damping very well for cases of small damping. And, the Q-factor method tends to over-estimate the damping ratio, as the true damping cannot be regarded as small. However, the present FCR method still predicts the result quite close to the true one, at lease with the confidence level of 90%, or 0.5 ∈ (0.482, 0.601). Refer to case 4 in Table 1.

Thus, it is evidently from the results of case 1 through 4 in Table 1 that the present FCR method can accurately identify the system damping with wide damping spectrum, from small to large damping. In fact, the present study has found that the method is very adequate for large damping systems, though it can be used to identify all structural damping as long as the damping is oscillatory. In addition, it has also been found that the frequency resolution is the critical factor to the present method. In case the frequency resolution is too high to locate the right resonance frequency, a large error may be incorporated. Nevertheless, the numerical simulation did substantiate the validity of the present FCR method for evaluation the system damping ratio.

4. Experiments

In addition to the numerical verification given in the last section, several experimental measurements have been carried out. The experimental set-up for the current study is shown in Fig. 4. The test specimen was a steel cantilever of dimensions $1.0^{T} \times 21.5^{W} \times 194^{L}$ mm (ca. 35.6 g) that was directly mounted on to a shaker. At the end of the beam, there was a plastic straw with the length 96 mm and the outer diameter 8.6 mm (ca. 2 g) firmly glued on to the beam. In the meantime, in order to simulate damping forces acting on the beam, the straw was inserted 20 mm into the liquid during the experiments. Refer to Fig. 4 the detail.

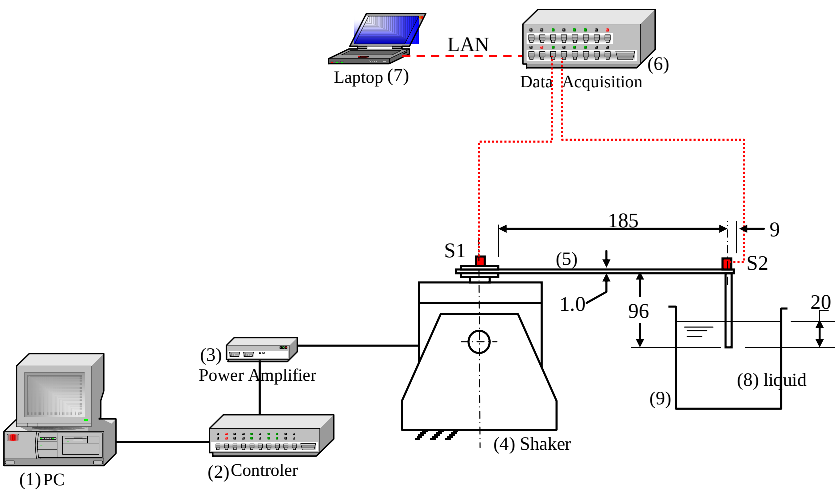

Figure 4. The experimental set-up (1: PC; 2: signal controller, Dactron; 3: Power Amplifier, B&K 2706; 4: Exciter, B&K 4809, 5: Cantilever; 6: Acquisition system, iMC μ-Musys; 7: PC and Software, FAMOS, 8: Liquid, and 9: Glass cup).

Theoretically, the beam has its undamped natural frequency at ca. 15.1 Hz obtained by CAE. However, it has been experimentally found that the maximum response amplitude appears at frequency (ω_m) 14.4 ± 0.05 Hz with 90% confidence level when the liquid is water. And, the corresponding mean quality factor Q is around 18.7, or ca. corresponding to ς = 2.69%.

During the all experiments, the sampling frequency f_s was set to 200 Hz. And, sinusoidal excitations with the frequency right at ω_m were applied from the controller to the shaker and detected by accelerometer 1 (S1), which directly mounted on the base of the cantilever; cf. Fig. 4. The system responses were then detected and acquired by accelerometer 2 (S2), which was located at the tip of the beam. The both signals were then acquired and sent to the laptop computer for the later analyses.

Figure 5 shows typical decaying responses in the time domain. It can be clearly seen from the figure that the decaying responses of the beam. Using the most commonly seen method, the logarithmic decrement [e.g., 9], one is able to estimate the system damping by the ratios of these decaying amplitudes. Accordingly, one computes this ratio and obtains the averaged system damping ratio is around 2.15%.

On the other hand, for the present FCR method, eq. (24) predicts the modulations containing both the time-invariant and signals of 2Ω. A typical result in the frequency domain is shown in Fig. 6. The results plotted in Fig. 6 obviously substantiate its validity. However, it can be also noticed that there exists a relatively small harmonics right at Ω. This can never theoretically happen according to eq. (24). The reason for this discrepancy stems mainly from the digital filter that was being applied during the data acquisition. This is also known as the signal leakage [5].

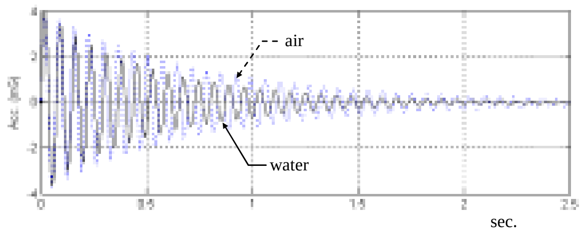

Figure 5 The decaying time responses of the beam.

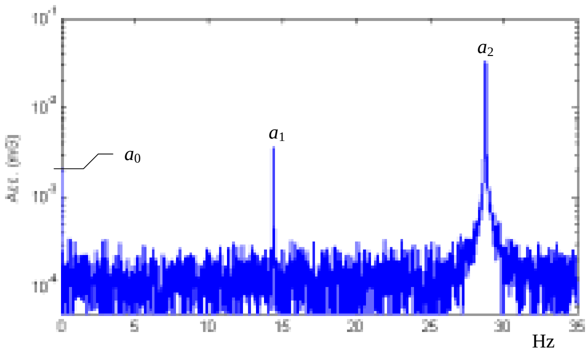

Figure 6 A typical modulated response in the frequency domain.

The measured results are tabulated in Table 2 for the three different methods. As one can read from the table that the present FCR method measures the system damping ratio with very good accuracy. In addition, this new method also the lag phase can be also computed. In fact, the angle due to the damping for the setting in Fig. 4 is 86.6° if the liquid is water for the example system.

5. Conclusions

A novel method for evaluating the linear viscous damping is given in this paper. The method is derived from the measurements of the system steady-state response and the input excitations. Using the input forcing function as the reference signals, the present method modulated the output steady-state responses. The modulated signals are then transformed into the frequency domain by the Fourier Transform. By doing so, one is able to identify the Fourier coefficients. It is the reason that the present method is named the Fourier Coefficient Ratio (FCR) method. The whole theoretical background has been systematically derived in the present paper. In addition, several simulations have been performed. Moreover, a cantilever beam with an end damper system has been designed to experimentally verify the novel method. The measured results from the FCR method have not only substantiated the validity of the new method, but with good accuracy as well. Unlike the most existing method, as it has been pointed out in the paper, the new FCR method can be applied to systems with a wider damping spectrum.

Table 2 The measured damping ratios

	Method	Mean ζ	Remarks
1	Q-Factor	2.69 %	water/20
2	Log. decrement	2.15%	water/20
3	FCR	2.10%	water/20

6. Acknowledgement

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